

# Note on antichain cutsets in discrete semimodular lattices

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## Abstract

*The characterization of level sets of finite Boolean lattices as antichain cutsets, due to Rival and Zaguia, is seen to hold in all discrete semimodular lattices.*

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## 1 Background

An *antichain cutset* in a partially ordered set is a set of elements intersecting every maximal chain in a singleton. Finite non-empty posets always have antichain cutsets. In *ranked posets with a least element* 0 (i.e. where in every interval  $[0, x]$  all maximal chains have the same finite number of elements  $\neq 0$ , called the *height*  $h(x)$  of  $x$ ), each set

$$L_n = \{x : h(x) = n\} \quad n = 0, 1, \dots$$

is an antichain cutset if  $L_n \neq \emptyset$ . An early study involving antichain cutsets is by Grillet [G]. Rival and Zaguia have shown in [RZ], Theorem 4, that in finite Boolean lattices height classes  $L_n$  are the only antichain cutsets. In the next section we shall see that this result extends to all discrete semimodular lattices. Such lattices may not have a least element, we therefore define, in any poset, a *level* as an equivalence class of the equivalence relation  $\equiv$  which is obtained as the reflexive-transitive closure of the following symmetric relation  $\sim$  :

$$x \sim y \quad \Leftrightarrow \quad \exists z \text{ covered by both } x \text{ and } y$$

For finite Boolean lattices this gives an alternative description of the sets  $L_n$  which remains meaningful in the larger context of discrete, possibly infinite and unbounded semimodular lattices. By *semimodularity* we understand the lower covering condition

$$x \text{ covers } x \wedge y \Rightarrow x \vee y \text{ covers } y$$

By a *discrete* order we mean a poset in which every interval  $[x, y]$  has a finite maximal chain. In a discrete semimodular lattice, for  $x \leq y$  all maximal chains of  $[x, y]$  have the same finite number of elements  $\neq x$ , called the *height of  $y$  above  $x$* , denoted  $h(x, y)$ .

It is easy to see that in finite semimodular lattices, which are ranked posets, levels defined as equivalence classes of the reflexive-transitive closure  $\equiv$  of the relation  $\sim$  coincide with the non-empty sets  $L_n = \{x : h(x) = n\}$ ,  $n = 0, 1, \dots$ . In all discrete semimodular lattices, two elements  $x, y$  are in the same level class if and only if they have the same height above their meet. In fact for any common lower bound  $z$  of  $x$  and  $y$ ,  $h(z, x) = h(z, y)$  if and only if  $x$  and  $y$  are in the same level class.

## 2 Generalized statement and proof

The following generalizes Theorem 4 of [RZ].

**Theorem** *Let  $L$  be any discrete semimodular lattice, and let  $A \subseteq L$ . Then  $A$  is an antichain cutset if and only if  $A$  is a level class.*

**Proof** For  $x < y$  we write  $h(y, x)$  for the negative of the height of  $y$  above  $x$ . Thus  $h(y, x) = -h(x, y)$  for all comparable  $x, y$ .

Suppose  $A$  is a level class. First, if  $x, y \in A$ , then  $h(x \wedge, x) = h(x \wedge y, y)$ , which rules out  $x < y$ . Thus  $A$  is an antichain. Second, let  $C$  be a maximal chain. Choose any  $a \in A$  and  $y \in C$ , and let  $z = a \wedge y$ . The chain  $C$  must contain an element  $x$  such that  $h(x, y) = h(z, y) - h(z, a)$ . Then  $a$  and  $x$  have the same height above  $z$  and therefore  $x$  is also in the level class  $A$ . This concludes the proof that  $A$  is an antichain cutset.

Suppose that  $A$  is an antichain cutset but not a level class. Choose any  $a \in A$  and let  $N$  be the level class of  $a$ . As  $N$  is an antichain cutset,  $N \not\subseteq A$ . Choose any  $b \in A \setminus N$ . There is a sequence of elements of  $N$ ,

$$x_0 = a, x_1, \dots, x_n = b$$

such that  $x_i \wedge x_{i+1}$  is covered by  $x_i$  and  $x_{i+1}$  for each  $i = 0, \dots, n-1$ . For the first index  $i$  with  $x_{i+1} \notin A$ , write

$$x = x_i, \ y = x_{i+1}, \ z = x \wedge y, \ w = x \vee y$$

As  $z$  is covered by  $x$  and  $y$ , by semimodularity  $w$  covers both  $x$  and  $y$ . Clearly the chain  $\{z, y, w\}$  avoids  $A$ . Let  $C$  be any maximal chain containing  $\{z, y, w\}$ . Then  $C$  must also avoid  $A$ , contradicting the assumption that  $A$  is an antichain cutset.  $\square$

## References

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- [RZ] I. Rival, N. Zaguia, Antichain cutsets, Order 1 (1985) 235-247